A26

MATHEMATICS
SUPPORT CENTRE

Title: Remainder Theorem and Factor Theorem

Target: On completion of this worksheet you should be able to use the remainder and factor theorems to find factors of polynomials.

Generally when a polynomial is divided by a linear expression there is a remainder.

e.g. $(3x^3 + 4x^2 - 5x + 3) \div (x + 2)$

$$\begin{array}{r l}
 & 3x^2 - 2x - 1 \\
x+2 & \overline{)\,3x^3 + 4x^2 - 5x + 3} \\
 & \underline{3x^3 + 6x^2} \\
 & \qquad -2x^2 - 5x \\
 & \qquad \underline{-2x^2 - 4x} \\
 & \qquad\qquad -x + 3 \\
 & \qquad\qquad \underline{-x - 2} \\
 & \qquad\qquad\quad 5
\end{array}$$

$$(3x^3 + 4x^2 - 5x + 3) = \underbrace{(x + 2)}_{\text{divisor}}\underbrace{(3x^2 - 2x - 1)}_{\text{quotient}} + \underbrace{5}_{\text{remainder}}$$

Any polynomial can be written in the following form:
polynomial ≡ divisor × quotient + remainder.

In particular if the divisor is $(x - a)$ and the polynomial is $f(x)$ then

$f(x) \equiv (x - a) \times$ quotient + remainder.

If $x = a$ then
$f(a) = (a - a) \times$ quotient + remainder.
$f(a)$ = remainder

This gives an easy way of finding the remainder when a polynomial is divided by $(x - a)$

Examples

1. Using previous example

$$f(x) = 3x^3 + 4x^2 - 5x + 3$$
$$= (x + 2)(3x^2 - 2x - 1) + 5$$

Now using $x = -2$

$$f(-2) = (-2 + 2)(3\times(-2)^2 - 2\times(-2) - 1) + 5$$
$$= 0\times(3\times(-2)^2 - 2\times(-2) - 1) + 5$$
$$= 5$$

i.e. the remainder.

2. Find the remainder when $(2x^3 - 5x^2 + x - 3)$ is divided by $(x - 1)$

Let $f(x) = 2x^3 - 5x^2 + x - 3$

Substitute $x = 1$ since we require $(x - 1) = 0$

$f(1) = 2\times 1^3 - 5\times 1^2 + 1 - 3 = -5$

The remainder is –5

3. Find the remainder when $(x^4 + 4x^2 + 3x - 7)$ is divided by $(x + 3)$

Let $f(x) = x^4 + 4x^2 + 3x - 7$

Substitute $x = -3$ since we require $(x + 3) = 0$

$f(-3) = (-3)^4 + 4\times(-3)^2 + 3\times(-3) - 7 = 101$

The remainder is 101

Exercise

Find the remainders for the following:

1. $(x^3 - 5x^2 + 6x - 4) \div (x - 2)$
2. $(4x^3 + 3x^2 + x + 2) \div (x - 1)$
3. $(2x^4 - x^3 + 3x^2 - 1) \div (x + 1)$
4. $(2x^3 - 6x - 5) \div (x + 3)$
5. $(x^3 - 4x^2 - x) \div (x - 4)$

(Answers: -4, 10, 5, -29, -4)

Example

Find the remainder when

$(3x^3+4x^2-5x-2)$ is divided by $(x-1)$

Let $f(x)=3x^3+4x^2-5x-2$ and $x=1$

$f(1)=3\times1^3+4\times1^2-5\times1-2=0$

The remainder is 0.

$$\begin{aligned}3x^3+4x^2-5x-2&=(x-1)\times\text{quotient}+0\\&=(x-1)\times\text{quotient}\end{aligned}$$

so $(x-1)$ is a factor of $(3x^3+4x^2-5x-2)$

We can use the remainder theorem to check for factors of a polynomial.

As before

$$f(x)=(x-a)\times\text{quotient}+\text{remainder}$$

and $\quad f(a)=\text{remainder}$

If $(x-a)$ is a factor then the remainder is 0

ie $f(a)=0$

This is called the factor theorem.

Examples

1. Is $(x-3)$ a factor of $(2x^3-3x^2-8x-3)$?

Let $f(x)=(2x^3-3x^2-8x-3)$ and $x=3$

as we are checking whether $(x-3)$ is a factor.

$f(3)=2\times3^3-3\times3^2-8\times3-3=0$

so $(x-3)$ is a factor of $(2x^3-3x^2-8x-3)$

2. Is $(x-1)$ a factor of $(2x^3-3x^2-8x-3)$?

Using $f(x)$ as above and $x=1$

$f(1)=2\times1^3-3\times1^2-8\times1-3=-12\neq0$

so $(x-1)$ is not a factor of $(2x^3-3x^2-8x-3)$

Exercise

1. Is $(x-1)$ a factor of

$$f(x)=(x^3+2x^2-2x-1)?$$

2. Is $(x+2)$ a factor of $f(x)=(4x^2+13x+10)$?
3. Is $(x-2)$ a factor of $f(x)=(4x^2+13x+10)$?
4. Is $(x+3)$ a factor of

$$f(x)=(3x^3+10x^2+x-6)?$$

5. Is $(x-1)$ a factor of

$$f(x)=(3x^3+10x^2+x-6)?$$

(Answers: yes, yes, no, yes, no)

We can use the factor theorem to factorise polynomials, although some trial and error is involved.

Example

Factorise $(2x^3+5x^2-x-6)$.

Let $f(x)=2x^3+5x^2-x-6$. Since the constant is –6 we will consider factors of this ie. ± 1, ± 2, ± 3, ± 6. We will try $(x-1)$

$f(1)=2\times1^3+5\times1^2-1-6=0$

so $(x-1)$ is a factor.

Now we can find the quadratic factor by division or by repeating the above.

$$\begin{array}{r}2x^2+7x+6\\(x-1)\overline{)2x^3+5x^2-x-6}\\\underline{2x^3-2x^2}\\7x^2-x\\\underline{7x^2-7x}\\6x-6\\\underline{6x-6}\\0\end{array}$$

$$\begin{aligned}f(x)&=2x^3+5x^2-x-6\\&=(x-1)(2x^2+7x+6)\\&=(x-1)(x+2)(2x+3)\end{aligned}$$

The quadratic factor is factorised in the normal way.

Exercise

Factorise the following:

1. $f(x)=x^3+2x^2-5x-6$
2. $f(x)=2x^3+x^2-2x-1$
3. $f(x)=x^3-3x^2-3x-4$
4. $f(x)=3x^3+6x^2+x+2$
5. $f(x)=4x^3-15x^2+17x-6$

Answers:

1. $(x+1)(x-2)(x+3)$
2. $(x+1)(x-1)(2x+1)$
3. $(x-4)(x^2+x+1)$
4. $(x+2)(3x^2+1)$
5. $(x-1)(x-2)(4x-3)$