

MATHEMATICS

SUPPORT CENTRE

Title: Chain Rule: further examples

Target: On completion of this worksheet you should be able to use the chain rule to differentiate further examples of functions of a function.

The chain rule

If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Examples

Differentiate the following:

1. $y = \sin(x^2 + 3)$

let $u = x^2 + 3$ so $y = \sin u$

$$\frac{du}{dx} = 2x \quad \frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = \cos u \times 2x$$

$$\frac{dy}{dx} = 2x \cos(x^2 + 3)$$

2. $y = \ln(3x^3 - 4x + 2)$

$u = 3x^3 - 4x + 2$ $y = \ln u$

$$\frac{du}{dx} = 9x^2 - 4 \quad \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{1}{u} \times (9x^2 - 4)$$

$$\frac{dy}{dx} = \frac{(9x^2 - 4)}{(3x^3 - 4x + 2)}$$

Note that $\frac{dy}{dx} = \frac{u'}{u}$

In general if

$$y = \ln f(x) \text{ then } \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

This is a useful result to remember.

Exercise

Differentiate the following:

1. $y = \cos(2x + 3)$

2. $y = \sin(4x - 1)$

3. $y = e^{(5x+4)}$

4. $y = \tan(x^3 + x)$

5. $y = \sin(4x^5 - 3x)$

6. $y = e^{2x^2}$

7. $y = \ln(3x + 2)$

8. $y = \cos(e^x)$

9. $y = \ln(\cos 4x)$

10. $y = \ln(x^2 - 3x + 4)$

11. $y = \ln(4x^5 - 3)$

12. $y = e^{\sin 5x}$

Answers:

1. $-2 \sin(2x + 3)$

2. $4 \cos(4x - 1)$

3. $5e^{(5x+4)}$

4. $(3x^2 + 1) \sec^2(x^3 + x)$

5. $(20x^4 - 3) \cos(4x^5 - 3x)$

6. $4xe^{2x^2}$ 7. $\frac{3}{3x+2}$

8. $-e^x \sin(e^x)$ 9. $\frac{-4 \sin 4x}{\cos 4x}$

10. $\frac{2x-3}{x^2-3x+4}$ 11. $\frac{20x^4}{4x^5-3}$

12. $(5 \cos 5x)e^{\sin 5x}$

Examples

Differentiate the following:

1. $y = \sin^2 4x$

Now $\sin^2 4x = (\sin 4x)^2$

let $u = \sin 4x$ $y = u^2$

$$\frac{du}{dx} = 4 \cos 4x \quad \frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = 2u \times 4 \cos 4x$$

$$\frac{dy}{dx} = 8 \sin 4x \cos 4x$$

2. $y = 2 \cos(3x - 7)^4$

$u = (3x - 7)^4$ $y = 2 \cos u$

To differentiate u we must use the chain rule

$v = 3x - 7$ $u = v^4$

$$\frac{dv}{dx} = 3 \quad \frac{du}{dv} = 4v^3$$

$$\frac{du}{dx} = 4v^3 \times 3 = 12v^3 = 12(3x - 7)^3$$

$$\frac{dy}{du} = -2 \sin u$$

$$\frac{dy}{dx} = -2 \sin u \times 12(3x - 7)^3$$

$$\frac{dy}{dx} = -24(3x - 7)^3 \sin(3x - 7)^4$$

3. $x = \frac{3}{\tan 5t}$

Now $x = 3(\tan 5t)^{-1}$

$u = \tan 5t$ $x = 3u^{-1}$

$$\frac{du}{dt} = 5 \sec^2 5t \quad \frac{dx}{du} = -1 \times 3u^{-2} = -\frac{3}{u^2}$$

$$\frac{dx}{dt} = -\frac{3}{u^2} \times 5 \sec^2 5t$$

$$\frac{dx}{dt} = -\frac{3}{\tan^2 5t} \times 5 \sec^2 5t = -\frac{\cos^2 5t}{\sin^2 5t} \times \frac{15}{\cos^2 5t}$$

$$\frac{dx}{dt} = -15 \operatorname{cosec}^2 5t$$

Exercise

1. $y = \cos^3 x$

2. $y = 3 \tan^2 2x$

3. $y = \frac{1}{\sin(2x+3)}$

4. $y = (e^{1-3x} + 3x)^4$

5. $s = 5 \sin^2(t-7) + 3 \cos^2 2t$

6. $p = 2 \ln(3-4q)^5$

7. $x = \ln(\sin t^2)$

8. $x = \ln(\sin^2 t)$

9. $y = \sqrt{\ln(5x+3)}$

10. $y = \frac{4}{\sqrt{\cos x^3}}$

Answers:

1. $-3 \cos^2 x \sin x$

2. $12 \tan 2x \sec^2 2x$

3. $\frac{-2 \cos(2x+3)}{\sin^2(2x+3)}$

4. $12(1 - e^{1-3x})(e^{1-3x} + 3x)^3$

5. $10 \sin(t-7) \cos(t-7) - 12 \sin 2t \cos 2t$

6. $\frac{-40}{(3-4q)^4}$

7. $\frac{2t \cos t^2}{\sin t^2} = 2t \cot t^2$

8. $\frac{2 \cos t}{\sin t} = 2 \cot t$

9. $\frac{5}{2(5x+3)\sqrt{\ln(5x+3)}}$

10. $\frac{6x^2 \sin x^3}{\cos^{\frac{3}{2}} x^3}$