C7

MATHEMATICS SUPPORT CENTRE

Title: Integration 1

Target: On completion of this worksheet you should be able to integrate simple functions.

Integration is the opposite of differentiation. If we know the derivative of a function then we can use integration to find the function.

We know that if

$$\frac{dy}{dx} = 2x \qquad \text{then} \qquad y = x^2$$

the power of x has increased by 1, to 2, and we have divided by 2

and if $\frac{dy}{dx} = 3x^2$ then $y = x^3$

again the power of x has increased by 1, to 3, and we have divided by 3.

In general if $\frac{dy}{dx} = x^n$ then $y = \frac{x^{n+1}}{n+1}$

but consider the following:

$$y = x^2 + 3 \qquad y = x^2 - 5 \qquad y = x^2 + 8$$

$$\frac{dy}{dx} = 2x \qquad \frac{dy}{dx} = 2x \qquad \frac{dy}{dx} = 2x$$

If there is a constant it disappears so

if $\frac{dy}{dx} = x^n$ then $y = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$

We call c the constant of integration.

Examples

1. $\frac{dy}{dx} = x^4 \qquad y = \frac{x^5}{5} + c$

2. $\frac{dy}{dx} = x^{-\frac{1}{2}} \qquad y = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$$y = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$y = 2x^{\frac{1}{2}} + c$$

Exercise

Find y:

1. $\frac{dy}{dx} = x^6$ 2. $\frac{dy}{dx} = x^{10}$ 3. $\frac{dy}{dx} = x^7$

4. $\frac{dy}{dx} = x^{-4}$ 5. $\frac{dy}{dx} = x^{-3}$ 6. $\frac{dy}{dx} = x^{-2}$

7. $\frac{dy}{dx} = x^{\frac{1}{2}}$ 8. $\frac{dy}{dx} = x^{\frac{3}{2}}$ 9. $\frac{dy}{dx} = x^{\frac{5}{4}}$

(Answers: $\frac{x^7}{7} + c, \frac{x^{11}}{11} + c, \frac{x^8}{8} + c, -\frac{x^{-3}}{3} + c, -\frac{x^{-2}}{2} + c, -\frac{1}{x} + c, \frac{2x^{\frac{3}{2}}}{3} + c, \frac{2x^{\frac{5}{2}}}{5} + c, \frac{4x^{\frac{9}{4}}}{9} + c$)

Notation

We use the notation $\int y\,dx$ to mean integrate y with respect to x.

Examples

1. $\int \sqrt[3]{x}\,dx = \int x^{\frac{1}{3}}\,dx$

$$= \frac{x^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} + c$$

$$= \frac{3x^{\frac{4}{3}}}{4} + c$$

2. $\int 6x^2\,dx = 6 \times \frac{x^3}{3} + c$

$$= 2x^3 + c$$

Note: the coefficient '6' is treated in the same way as when differentiating ie. multiply the integrated function by 6.

Exercise

1. $\int \sqrt{x}\,dx$
2. $\int 4x\,dx$
3. $\int 5x^9\,dx$
4. $\int \tfrac{1}{2}x^{-5}\,dx$

(Answers : $\dfrac{2x^{\frac{3}{2}}}{3}+c, 2x^2+c, \dfrac{x^{10}}{2}+c, -\tfrac{1}{8}x^{-4}+c$)

In general:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1}+c \qquad n \neq -1$$

We can integrate sums and differences of functions term by term just as we do when differentiating.

Examples

1. $\int (2x^3+5x-1)dx = \dfrac{2x^4}{4}+\dfrac{5x^2}{2}-x+c$

$$= \frac{x^4}{2}+\frac{5x^2}{2}-x+c$$

2. $\int (x-5)(3x+1)dx$

First we must multiply out the brackets and then integrate term by term:

$$\int (x-5)(3x+1)dx = \int 3x^2+x-15x-5dx$$
$$= \int 3x^2-14x-5dx$$
$$= \frac{3x^3}{3}-\frac{14x^2}{2}-5x+c$$
$$= x^3-7x^2-5x+c$$

Exercise

1. $\int (7x-3)dx$
2. $\int (3x^5+2x-8)dx$
3. $\int (\tfrac{1}{3}x^2+4x^{-2}+x-1)dx$
4. $\int 2x(3x-5)dx$ (Multiply out brackets first)
5. $\int (\sqrt[4]{x}+\dfrac{5}{x^2})dx$

(Answers : constant of integration is omitted

$\dfrac{7x^2}{2}-3x,\ \tfrac{1}{6}x^6+x^2-8x,\ \tfrac{1}{9}x^3-4x^{-1}+\tfrac{1}{2}x^2-x$

$2x^3-5x^2,\ \tfrac{4}{5}x^{\frac{5}{4}}-5x^{-1}$)

Other standard integrals are:

$$\int \sin ax\,dx = -\tfrac{1}{a}\cos ax+c$$
$$\int \cos ax\,dx = \tfrac{1}{a}\sin ax+c$$
$$\int \sec^2 ax\,dx = \tfrac{1}{a}\tan ax+c$$
$$\int e^{ax}dx = \tfrac{1}{a}e^{ax}+c$$
$$\int \frac{1}{ax}dx = \tfrac{1}{a}\ln|x|+c$$

($|x|$ means the modulus of x ie the absolute value)

Examples

1. $\int 2\cos 3x\,dx = 2\times\tfrac{1}{3}\sin 3x+c$

$$= \tfrac{2}{3}\sin 3x+c$$

2. $\int (3\sin x+4e^{2x})dx = 3\times(-\cos x)+4\times\tfrac{1}{2}e^{2x}+c$

$$= 3\cos x+2e^{2x}+c$$

Exercise

1. $\int 3\cos 2x\,dx$
2. $\int 2\sin 4x+3\cos 5x\,dx$
3. $\int 4e^{-x}+3e^{2x}dx$
4. $\int \dfrac{2}{x}+3x^2-4x+6dx$
5. $\int 7\sec^2 2x+3\sin 2x\,dx$
6. $\int \dfrac{2x^2+3}{x}dx$
7. $\int 4\sin\left(\dfrac{x}{2}\right)+5\cos\left(\dfrac{x}{2}\right)dx$
8. $\int (\tfrac{3}{2}\cos 4x-\tfrac{1}{4}\sin 4x)dx$
9. $\int 2e^{-x}(3e^{2x}+5e^{x})dx$
10. $\int \left(\dfrac{e^{2x}-e^{-2x}}{2}\right)dx$

(Answers :constant of integration is omitted

$\tfrac{3}{2}\sin 2x, \tfrac{3}{5}\sin 5x-\tfrac{1}{2}\cos 4x, \tfrac{3}{2}e^{2x}-4e^{-x}$

$2\ln|x|+x^3-2x^2+6x, \tfrac{7}{2}\tan 2x-\tfrac{3}{2}\cos 2x,$

$x^2+3\ln|x|,\ 10\sin\tfrac{x}{2}-8\cos\tfrac{x}{2}, \tfrac{3}{8}\sin 4x+\tfrac{1}{10}\cos 4x,$

$6e^x+10x\tfrac{1}{4}, e^{2x}+\tfrac{1}{4}e^{-2x}$)