C8

MATHEMATICS SUPPORT CENTRE

Title: Definite Integration

Target: On completion of this worksheet you should be able to evaluate definite integrals and areas under curves.

Definition

If $\int f(x)dx = F(x) + c$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

This process is called 'definite integration'.
The procedure is:

- integrate the function
- substitute the upper limit (*b*)
- substitute the lower limit (*a*)
- subtract

(Note that the constant of integration is left out as it will disappear when the two parts are subtracted.)

Examples

1. $$\int_2^4 3x+5dx = \left[\frac{3x^2}{2}+5x\right]_2^4$$
$$= \left(\frac{3\times 4^2}{2}+5\times 4\right)-\left(\frac{3\times 2^2}{2}+5\times 2\right)$$
$$= 44-16$$
$$= 28$$

2. $$\int_0^1 (2\cos 3x - \sin 3x)dx = \left[\frac{2\sin 3x}{3}+\frac{\cos 3x}{3}\right]_0^1$$
$$= \left(\frac{2\sin 3}{3}+\frac{\cos 3}{3}\right)-\left(\frac{2\sin 0}{3}+\frac{\cos 0}{3}\right)$$
$$= (-0\cdot 2359)-0\cdot 3333$$
$$= -0\cdot 5692$$
$$= -0\cdot 569$$

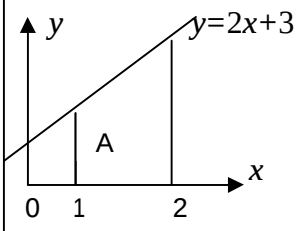
Remember to use radians when integrating (or differentiating) trigonometric functions.

Exercise

1. $\int_0^3 x^2 - 3x + 7dx$
2. $\int_1^5 4x^3 + 6x + 1dx$
3. $\int_{-2}^2 (2x-3)(x+1)dx$
4. $\int_1^5 \frac{2x^3 + x^2 - 3}{x^2}dx$
5. $\int_0^2 3e^x + 4e^{-x}dx$
6. $\int_0^{\frac{p}{3}} 7\sec^2 2tdt$
7. $\int_{10}^{20} \sqrt[3]{x} + 4\sqrt{x} - \frac{3}{x}dx$
8. $\int_2^5 (3\cos 2x + 7\sin x + 5x - 1)dx$
9. $\int_0^1 2e^{-2x}(e^{3x} + e^{-3x})dx$
10. $\int_{-\frac{p}{2}}^{\frac{p}{2}} \sin\frac{x}{3} + 2\cos\frac{2x}{3}dx$

(Answers: 16·5, 700, -1·33, 25·6, 22·6, -6·06 177, 42·4, 3·83, 5·20)

Consider the function $y = 2x + 3$. Suppose we want to find the area enclosed by the line, the horizontal axis and the two vertical lines at $x = 1$ and $x = 2$:



The trapezium A is the required area. When $x = 1$, $y = 5$ and when $x = 2$, $y = 7$. These are the lengths of the parallel sides:

area = ½(5 + 7)×1
= 6

Now we will evaluate the definite integral

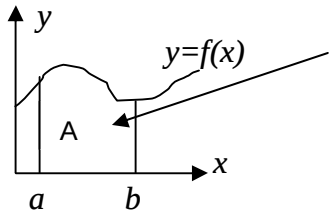
$$\int_1^2 2x+3\,dx = \left[\frac{2x^2}{2}+3x\right]_1^2$$
$$= \left[x^2+3x\right]_1^2$$
$$= \left(2^2+3\times 2\right)-\left(1^2+3\times 1\right)$$
$$= 6$$

area A = $\int_1^2 2x+3\,dx$

This is true in general:

If $y = f(x)$ then the area bounded by the curve, the x-axis and the lines $x = a$ and $x = b$ is

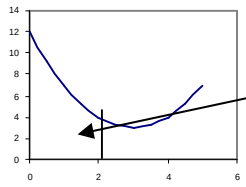
$$\int_a^b f(x)\,dx$$



area A is $\int_a^b f(x)\,dx$

Examples

1. Find the area bounded by the curve $y = x^2 - 6x + 12$ the x-axis and the lines $x = 0$ and $x = 2$.

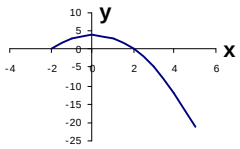


This is the required area

$$\int_0^2 (x^2-6x+12)\,dx = \left[\frac{x^3}{3}-\frac{6x^2}{2}+12x\right]_0^2$$
$$= (\tfrac{8}{3}-3\times 2^2+12\times 2)-(0)$$
$$= 14\tfrac{2}{3}$$

Examples cont.

2. Find the area bounded by the curve $y = 4 - x^2$, the x-axis and the lines $x = 1$ and $x = 4$



The area required is divided into two parts: one part, between $x = 1$ and $x = 2$, above the x-axis, and one part between $x = 2$ and $x = 4$ below the x-axis. We must find each area separately.

$$\text{First area} = \int_1^2 4-x^2\,dx$$
$$= \left[4x-\tfrac{1}{3}x^3\right]_1^2$$
$$= \left(4\times 2-\tfrac{1}{3}\times 2^3\right)-\left(4\times 1-\tfrac{1}{3}\times 1^3\right)$$
$$= \tfrac{5}{3}$$

$$\text{Second area} = \int_2^4 4-x^2\,dx$$
$$= \left[4x-\tfrac{1}{3}x^3\right]_2^4$$
$$= \left(4\times 4-\tfrac{1}{3}\times 4^3\right)-\left(4\times 2-\tfrac{1}{3}\times 2^3\right)$$
$$= -\tfrac{32}{3}$$

The negative sign indicates that the area is below the x-axis (see diagram above) so to find the total area we ignore the sign.

$$\text{Total area} = \tfrac{5}{3}+\tfrac{32}{3}$$
$$= 12\tfrac{1}{3}$$

Remember to draw a sketch of the graph first in case part of the area is below the x-axis. In this case calculate the areas separately and then add the answers, ignoring the minus sign.

Exercise

For each of the following functions find the area bounded by the curve, the x-axis and the given lines:

1. $y = x^2 + 3$ $\quad x = 0, x = 1$
2. $y = 2x^3$ $\quad x = -2, x = 2$
3. $y = (x+3)(x-2)$ $\quad x = 1, x = 4$
4. $y = e^x$ $\quad x = 0, x = 2$
5. $y = \sin x$ $\quad x = 1, x = 5$

(Answers: 3·33, 16, 14·8, 6·39, 2·82)