The double angle formulae

This unit looks at trigonometric formulae known as the double angle formulae. They are called this because they involve trigonometric functions of double angles, i.e. \( \sin 2A \), \( \cos 2A \) and \( \tan 2A \).

In order to master the techniques explained here it is vital that you undertake the practice exercises provided.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- derive the double angle formulae from the addition formulae
- write the formula for \( \cos 2A \) in alternative forms
- use the formulae to write trigonometric expressions in different forms
- use the formulae in the solution of trigonometric equations

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1. Introduction
This unit looks at trigonometric formulae known as the double angle formulae. They are called this because they involve trigonometric functions of double angles, i.e. \(\sin 2A\), \(\cos 2A\) and \(\tan 2A\).

2. The double angle formulae for \(\sin 2A\), \(\cos 2A\) and \(\tan 2A\)
We start by recalling the addition formulae which have already been described in the unit of the same name.

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]
\[
\cos(A + B) = \cos A \cos B - \sin A \sin B
\]
\[
\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}
\]

We consider what happens if we let \(B\) equal to \(A\). Then the first of these formulae becomes:

\[
\sin(A + A) = \sin A \cos A + \cos A \sin A
\]

so that

\[
\sin 2A = 2 \sin A \cos A
\]

This is our first double-angle formula, so called because we are doubling the angle (as in \(2A\)).

Similarly, if we put \(B\) equal to \(A\) in the second addition formula we have

\[
\cos(A + A) = \cos A \cos A - \sin A \sin A
\]

so that

\[
\cos 2A = \cos^2 A - \sin^2 A
\]

and this is our second double angle formula.

Similarly

\[
\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}
\]

so that

\[
\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}
\]

These three double angle formulae should be learnt.

---

**Key Point**

\[
\sin 2A = 2 \sin A \cos A \quad \cos 2A = \cos^2 A - \sin^2 A \quad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}
\]
3. The formula \( \cos 2A = \cos^2 A - \sin^2 A \)

We now examine this formula more closely.

We know from an important trigonometric identity that

\[
\cos^2 A + \sin^2 A = 1
\]

so that by rearrangement

\[
\sin^2 A = 1 - \cos^2 A.
\]

So using this result we can replace the term \( \sin^2 A \) in the double angle formula. This gives

\[
\begin{align*}
\cos 2A &= \cos^2 A - \sin^2 A \\
&= \cos^2 A - (1 - \cos^2 A) \\
&= 2 \cos^2 A - 1
\end{align*}
\]

This is another double angle formula for \( \cos 2A \).

Alternatively we could replace the term \( \cos^2 A \) by \( 1 - \sin^2 A \) which gives rise to:

\[
\begin{align*}
\cos 2A &= \cos^2 A - \sin^2 A \\
&= (1 - \sin^2 A) - \sin^2 A \\
&= 1 - 2 \sin^2 A
\end{align*}
\]

which is yet a third form.

---

**Key Point**

\[
\begin{align*}
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2 \cos^2 A - 1 \\
&= 1 - 2 \sin^2 A
\end{align*}
\]

---

4. Finding \( \sin 3x \) in terms of \( \sin x \)

**Example**

Consider the expression \( \sin 3x \). We will use the addition formulae and double angle formulae to write this in a different form using only terms involving \( \sin x \) and its powers.

We begin by thinking of \( 3x \) as \( 2x + x \) and then using an addition formula:
\[
\sin 3x = \sin(2x + x)
\]
\[
= \sin 2x \cos x + \cos 2x \sin x
\]
using the first addition formula
\[
= (2 \sin x \cos x) \cos x + (1 - 2 \sin^2 x) \sin x
\]
using the double angle formula
\[
= \cos 2x = 1 - 2 \sin^2 x
\]
\[
= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x
\]
\[
= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x
\]
from the identity \(\cos^2 x + \sin^2 x = 1\)
\[
= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x
\]
\[
= 3 \sin x - 4 \sin^3 x
\]
We have derived another identity
\[
\sin 3x = 3 \sin x - 4 \sin^3 x
\]
Note that by using these formulae we have written \(\sin 3x\) in terms of \(\sin x\) (and its powers). You could carry out a similar exercise to write \(\cos 3x\) in terms of \(\cos x\).

### 5. Using the formulae to solve an equation

**Example**

Suppose we wish to solve the equation \(\cos 2x = \sin x\), for values of \(x\) in the interval \(-\pi \leq x < \pi\).

We would like to try to write this equation so that it involves just one trigonometric function, in this case \(\sin x\). To do this we will use the double angle formula

\[
\cos 2x = 1 - 2 \sin^2 x
\]

The given equation becomes

\[
1 - 2 \sin^2 x = \sin x
\]

which can be rewritten as

\[
0 = 2 \sin^2 x + \sin x - 1
\]

This is a quadratic equation in the variable \(\sin x\). It factorises as follows:

\[
0 = (2 \sin x - 1)(\sin x + 1)
\]

It follows that one or both of these brackets must be zero:

\[
2 \sin x - 1 = 0 \quad \text{or} \quad \sin x + 1 = 0
\]

so that

\[
\sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1
\]

We can solve these two equations by referring to the graph of \(\sin x\) over the interval \(-\pi \leq x < \pi\) which is shown in Figure 1.

\[
\text{Figure 1. A graph of } \sin x \text{ over the interval } -\pi \leq x < \pi.
\]
From the graph we see that the angle whose sine is $-1$ is $-\frac{\pi}{2}$. The angle whose sine is $\frac{1}{2}$ is a standard result, namely $\frac{\pi}{6}$, or $30^\circ$. Using the graph, and making use of symmetry we note there is another solution at $x = \frac{5\pi}{6}$. So, in summary, the solutions are

$$x = \frac{\pi}{6}, \ \frac{5\pi}{6} \ \text{and} \ -\frac{\pi}{2}$$

**Example**

Suppose we wish to solve the equation

$$\sin 2x = \sin x \ \pi \leq x < \pi$$

In this case we will use the double angle formulae $\sin 2x = 2 \sin x \cos x$. This gives

$$2 \sin x \cos x = \sin x$$

We rearrange this and factorise as follows:

$$2 \sin x \cos x - \sin x = 0$$

$$\sin x(2 \cos x - 1) = 0$$

from which

$$\sin x = 0 \ \text{or} \ \cos x = \frac{1}{2}$$

We have reduced the given equation to two simpler equations. We deal first with $\sin x = 0$. By referring to the graph of $\sin x$ in Figure 1 we see that the two required solutions are $x = -\pi$ and $x = 0$. The potential solution at $x = \pi$ is excluded because it is outside the interval specified in the original question.

The equation $2 \cos x - 1 = 0$ gives $\cos x = \frac{1}{2}$. The angle whose cosine is $\frac{1}{2}$ is $60^\circ$ or $\frac{\pi}{3}$, another standard result. By referring to the graph of $\cos x$ shown in Figure 2 we deduce that the solutions are $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$.

![A graph of $\cos x$ over the interval $-\pi \leq x < \pi$.](image)

**Exercises**

1. Verify the three double angle formulae (for $\sin 2A$, $\cos 2A$, $\tan 2A$) for the cases $A = 30^\circ$ and $A = 45^\circ$.

2. By writing $\cos(3x) = \cos(2x + x)$ determine a formula for $\cos(3x)$ in terms of $\cos x$.  

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3. Determine a formula for \( \cos(4x) \) in terms of \( \cos x \).

4. Solve the equation \( \sin 2x = \cos x \) for \( -\pi \leq x < \pi \).

5. Solve the equation \( \cos 2x = \cos x \) for \( 0 \leq x < \pi \)

\[\begin{align*}
2. & \quad 4 \cos^3 x - 3 \cos x \\
3. & \quad 8 \cos^4 x - 8 \cos^2 x + 1 \\
4. & \quad \frac{-\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6} \\
5. & \quad 0 \text{ and } \frac{2\pi}{3}
\end{align*}\]