Differentiating logarithm and exponential functions

This unit gives details of how logarithmic functions and exponential functions are differentiated from first principles.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate $\ln x$ from first principles
- differentiate e^x

Contents

1. Introduction

In this unit we explain how to differentiate the functions $\ln x$ and e^x from first principles.
To understand what follows we need to use the result that the exponential constant e is defined as the limit as t tends to zero of $(1+t)^{1/t}$ i.e. $\lim_{t \to 0} (1+t)^{1/t}$.

To get a feel for why this is so, we have evaluated the expression $(1+t)^{1/t}$ for a number of decreasing values of t as shown in Table 1. Note that as t gets closer to zero, the value of the expression gets closer to the value of the exponential constant e≈ 2.718.... You should verify some of the values in the Table, and explore what happens as t reduces further.

t	$(1+t)^{1/t}$	
1	$(1+1)^{1/1}$	$= 2$
0.1	$(1+0.1)^{1/0.1}$	$= 2.594$
0.01	$(1+0.01)^{1/0.01}$	$= 2.705$
0.001	$(1.001)^{1/0.001}$	$= 2.717$
0.0001	$(1.0001)^{1/0.0001}$	$= 2.718$

We will also make frequent use of the laws of indices and the laws of logarithms, which should be revised if necessary.

2. Differentiation of a function $f(x)$

Recall that to differentiate any function, $f(x)$, from first principles we find the slope, $\frac{\delta y}{\delta x}$, of the line joining an arbitrary point, A, and a neighbouring point, B, on the graph of $f(x)$. We then determine what happens to $\frac{\delta y}{\delta x}$ in the limit as δx tends to zero. (See Figure 1).

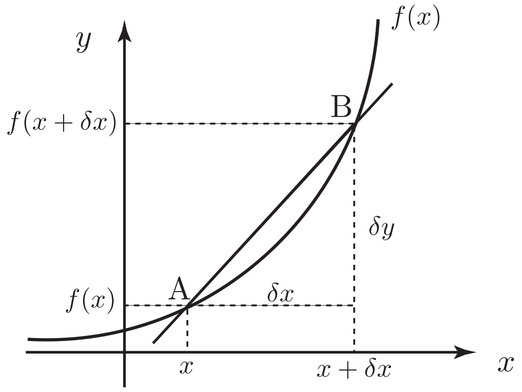

Figure 1. $\frac{\delta y}{\delta x}$ is the slope of AB.

The derivative, $f'(x)$, is then given by

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Use of this result has been explained at some length in the first unit on differentiation from first principles.

3. Differentiation of $f(x) = \ln x$

Using the definition of the derivative in the case when $f(x) = \ln x$ we find

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{\ln(x+\delta x) - \ln x}{\delta x}$$

We proceed by using the law of logarithms $\log A - \log B = \log \dfrac{A}{B}$ to re-write the right-hand side as firstly

$$\frac{1}{\delta x}\left(\ln(x+\delta x) - \ln x\right) = \frac{1}{\delta x}\ln\left(\frac{x+\delta x}{x}\right) = \frac{1}{\delta x}\ln\left(1+\frac{\delta x}{x}\right)$$

In order to simplify what will follow we make a substitution: let $t = \dfrac{\delta x}{x}$, that is, $\delta x = xt$. (This substitution is made because in the calculations which follow it is the ratio of δx to x which turns out to be important. We need not worry about x being zero because we are interested in differentiating $\ln x$ and the logarithm function is only defined for positive values of x.)

Then

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{1}{xt}\ln(1+t)$$

Further, using the law $n \log A = \log A^n$ we can take the $\dfrac{1}{t}$ inside the logarithm to give

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{1}{x}\ln(1+t)^{\frac{1}{t}}$$

Referring to the general case in Figure 1, this represents the slope of the line joining the two points on the graph of $f(x)$. To find the derivative we need to let δx tend to zero. Because we substituted $t = \dfrac{\delta x}{x}$ we need to let t tend to zero.

We have

$$f'(x) = \lim_{t \to 0} \frac{1}{x}\ln\left(1+t\right)^{\frac{1}{t}}$$

In this limiting process it is t which tends to zero, and we can regard x as a fixed number. So, it can be taken outside the limit to give:

$$f'(x) = \frac{1}{x}\lim_{t \to 0} \ln\left(1+t\right)^{\frac{1}{t}}$$

But we know that

$$\lim_{t \to 0}\left(1+t\right)^{\frac{1}{t}} = \mathrm{e}$$

and so

$$f'(x) = \frac{1}{x}\ln \mathrm{e} = \frac{1}{x}$$

since $\ln \mathrm{e} = 1$.

We have shown, from first principles, that the derivative of $\ln x$ is equal to $\dfrac{1}{x}$.

Key Point

$$\text{if} \quad f(x) = \ln x \quad \text{then} \quad f'(x) = \frac{1}{x}$$

Exercise

1. Show from first principles, using exactly the same technique, that if $f(x) = \log_{10} x$ then $f'(x) = \dfrac{1}{x \ln 10}$.

2. Show from first principles that if $f(x) = \log_a x$ then $f'(x) = \dfrac{1}{x \ln a}$.

4. Differentiation of $f(x) = \mathrm{e}^x$

To differentiate $y = \mathrm{e}^x$ we will rewrite this expression in its alternative form using logarithms:

$$\ln y = x$$

Then differentiating both sides with respect to x,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln y) = 1$$

The idea is now to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Recall that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln y) = \dfrac{\mathrm{d}}{\mathrm{d}y}(\ln y) \times \dfrac{\mathrm{d}y}{\mathrm{d}x}$. (This result is obtained using a technique known as **the chain rule**. You should refer to the unit on the chain rule if necessary).

Now we know, from Section 3, that $\dfrac{\mathrm{d}}{\mathrm{d}y}(\ln y) = \dfrac{1}{y}$ and so

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

Rearranging,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

But $y = \mathrm{e}^x$ and so we have the important and well-known result that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$$

Key Point

$$\text{if} \quad f(x) = \mathrm{e}^x \quad \text{then} \quad f'(x) = \mathrm{e}^x$$

The exponential function (and multiples of it) is the only function which is equal to its derivative.

Exercise

1. Show from first principles, using exactly the same technique, that if $f(x) = a^x$ then $f'(x) = a^x \ln a$.

